angle is only 5 degrees it is sufficient to account for the presence of higher order propagating waves at wavelengths noticeably longer than the theoretical $\lambda_0 = b$. In a typical case the higher waves were first noticed at $\lambda_0 = 9.8$ cm compared to 9.5 cm at which point they caused considerable distortion.

The calculated curves in Figs. 7–13 were obtained from the variational expressions using the fourth approximation. An estimate of the error showed the magnitude of the reflection coefficient to be within 1 per cent and the phase within 2 degrees.

Conclusion

The experimental results given are typical of a large number taken from about ten specimens and the agreement between theory and experiment is of the same order throughout. Allowing for the inherent error of the transmission line the worst error is −5 per cent for the magnitude of reflection coefficient and within 5 degrees for the phase. Apart from the region of interference mentioned above, the thick plate theory appears to be quite satisfactory. Departures from this theory are believed to be due to the difference between the ideal and practical situations, mainly in regard to the imperfect means of plane wave excitation. Further study of the higher order propagating waves is required, although in general it is not desirable to have these higher orders present. It is worth noting that for a constant refractive index, the reflection at a single interface can actually be reduced by intentionally increasing the plate thickness, up to the point at which the first higher order wave starts to propagate.

The solution for arbitrary plane wave incidence offers no fundamental difficulty and is being extended.

The Characteristic Impedance of Trough and Slab Lines*

ROBIN M. CHISHOLM†

Summary—**A variational method is used to develop an expression for the characteristic impedance of a "trough line" consisting of a circular cylinder mounted inside and parallel to the walls of a semi-infinite rectangular trough. The "slab line" consisting of a circular cylinder between infinite, parallel plates is treated as a special case of the trough line in which the bottom of the trough is taken to be infinitely remote from the circular cylinder. The solution has not been restricted to cylinders that are mounted exactly half way between the parallel walls of the trough; a simple formula is presented for calculating the tolerances which must be placed on the "centering" of the center conductor for a given allowable error in the characteristic impedance.**

The expression for the characteristic impedance is presented as the sum of three terms. The first is a "zero order" logarithmic term, the second a "second order" correction term which vanishes as the center conductor becomes infinitely small, and the third is an "off-center" correction term which arises when the cylinder is not exactly half way between the parallel walls of the trough. The second order correction term amounts to about 0.3 ohms when the characteristic impedance is of the order of 50 ohms. A fourth order approximation using the same method changes this by about 0.001 ohm.

Introduction

DIFFICULTIES in manufacturing slotted lines for coaxial systems have led to the investigation of special types of coaxial lines for this purpose. The present work is concerned with finding the characteristic impedance of two special types of two-conductor transmission lines which can be used for standing wave measurements. The "trough line," illustrated in Fig. 1, consists of a circular cylinder mounted inside and parallel to the walls of a rectangular trough. The "slab line," consisting of a circular cylinder between infinite, parallel planes, can be considered as a special case of the trough line in which the bottom of the trough is infinitely remote from the circular cylinder. The line is excited in the TEM mode by a generator connected between the circular cylinder and the walls of the trough and propagation is along the axis of the cylinder.

The practical difficulties involved in constructing coaxial slotted lines and the application of "slab lines" to the problem have been discussed in a paper by Wholey and Eldred.[1] These authors developed a solution for the "slab line" using conformal mapping to match the outer conductor everywhere and the inner conductor at four points which was accurate to about 0.1 ohm for characteristic impedances of the order of fifty ohms. Frankel[2] treated both the "trough line" and the "slab line" using conformal mapping for the case of an infinitely thin center conductor and, although a different method is

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[1] W. B. Wholey and W. N. Eldred, "A new type of slotted line section," Proc. IRE, vol. 38, pp. 244–248; March, 1950.

[2] S. Frankel, "Characteristic impedance of parallel wires in rectangular troughs," Proc. IRE, vol. 30, pp. 182–190; April, 1942.

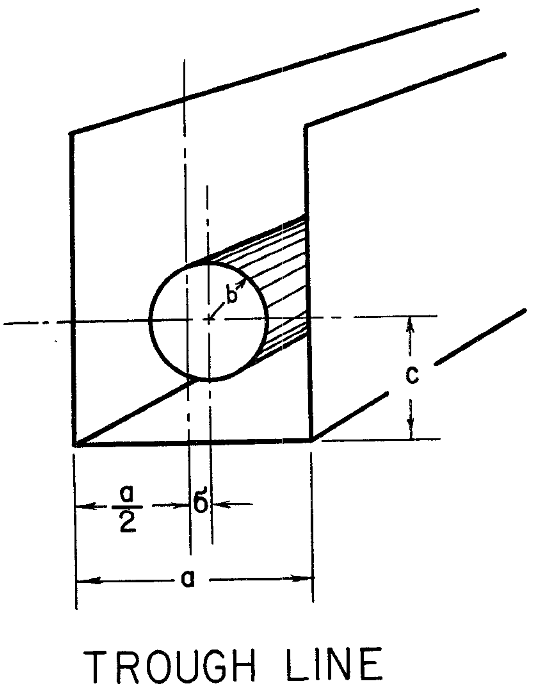

Fig. 1—The trough line.

used, the present work yields Frankel's expressions plus a correction term which vanishes for small wires. Wheeler[3] has recently treated the "slab line" using a pair of line sources to represent his assumed field varying their displacement in order to match the boundaries.

In the present work the "trough line" is treated directly by a variational method with the "slab line" taken as a special case. The solution has not been restricted to lines in which the center conductor is placed exactly half way between the parallel walls of the trough. The final result for the characteristic impedance is presented as the sum of three terms, the first being the "zero order" term of Frankel, the second a "second order" correction term which vanishes as the center conductor becomes small, and the third an "off-center" term which is proportional to the square of the distance the center cylinder is from the center line between the parallel planes.

Formulation of the Problem

It is a well known result of wave theory that a lossless, coaxial structure of uniform cross section propagates all frequencies at the velocity of light and the electric field at any cross section at a given instant is identical in form to the electrostatic field that would occur if a static charge were placed on the center conductor. The characteristic impedance and velocity of propagation of any uniform, two conductor, lossless transmission line are given by

[3] H. A. Wheeler, "The transmission properties of a round wire between parallel planes," Wheeler Monographs No. 19, Wheeler Laboratories Inc., Great Neck, N.Y.; 1954.

$$Z_0 = (L/C)^{1/2} \text{ ohms} \tag{1}$$

$$v = 1/(LC)^{1/2} \text{ meters per second} \tag{2}$$

where L is the inductance of the line in henries per meter and C is the capacitance of the line in farads per meter. The inductance L can be eliminated between (1) and (2) giving

$$Z_0 = 1/(vC); \quad v = 2.99796 \times 10^8 \text{ m/sec.} \tag{3}$$

The capacitance can then be found from the ratio of electrostatic charge on the center conductor to the voltage between conductors which would result from this charge. Let Q be the total charge per meter on the center conductor which is distributed around the periphery with a surface density $\rho(\phi)$ so that

$$Q = \int_0^{2\pi} \rho(\phi) b d(\phi) \tag{4}$$

where b is the radius of the center conductor as shown in Fig. 3.

The electrostatic field can be expressed in terms of a Green's function $G(r', \phi', r, \phi,)$ for the region. This function can be looked upon as the potential at point r, ϕ due to unit line charge at r', ϕ' [and of course subject to the boundary conditions that $G(r', \phi', r, \phi) = 0$ if r, ϕ is a point on the wall of the trough]. Using this representation the electrostatic field is given by the potential function

$$V(r, \phi) = (1/\epsilon) \int_0^{2\pi} \rho(\phi') G(b, \phi', r, \phi) b d\phi' \tag{5}$$

since the total charge in the region lies on the periphery of the cylinder where $r' = b$.

The charge distribution $\rho(\phi)$ is still unknown but is subject to the restriction that, for points on the cylinder ($r = b$), the potential must reduce to a constant value for all ϕ. This leads to an integral equation for $\rho(\phi)$ of the form

$$V_0 = (1/\epsilon) \int_0^{2\pi} \rho(\phi') G(b, \phi', b, \phi) b d\phi' \tag{6}$$

which can be solved using a variational method (Appendix I) to give a very rapidly converging expression for the capacitance C.

The Green's Function

The function $G(r, \phi, r', \phi')$, which is interpreted as the electrostatic potential in the region due to unit line charge, must obey the same boundary conditions as the desired potential function $V(r, \phi)$. These are, namely, that it vanish on the walls of the trough and at large distances to the left of the page in Fig. 3. In addition the Green's function must be symmetric in the primed and unprimed coordinates. It is a simple matter to verify that if $G(r', \phi', r, \phi)$ in (5) obeys

$$\nabla^2 G(r', \phi', r, \phi) = -\delta(\phi - \phi') \quad (7)$$

where ∇^2 is the Laplacian operator in two dimensions and $\delta(r-r')$ is the Dirac delta function, then V as given by (5) obeys the proper Poisson equation for the problem.

The function which obeys (7) and the proper boundary conditions can be readily found[4] and is expressible in rectangular coordinates as

$$G(x', y', x, y) = \frac{1}{2\pi}\left[\log_e \frac{1 - \exp\left(i(\pi/a)(y+y') - (\pi/a)\,|\,x-x'\,|\right)}{1 - \exp\left(i(\pi/a)(y-y') - (\pi/a)\,|\,x-x'\,|\right)} - \log_e \frac{1 - \exp\left(i(\pi/a)(y+y') - (\pi/a)(x+x'-2c)\right)}{1 - \exp\left(i(\pi/a)(y-y') - (\pi/a)(x+x'-2c)\right)}\right] \quad (8)$$

In this formula a is the distance between the parallel plates of the trough (see Fig. 2) and c is the distance from the center of the cylinder to the bottom of the trough. For the "slab line" c becomes infinite and the second term of (8) approaches zero.

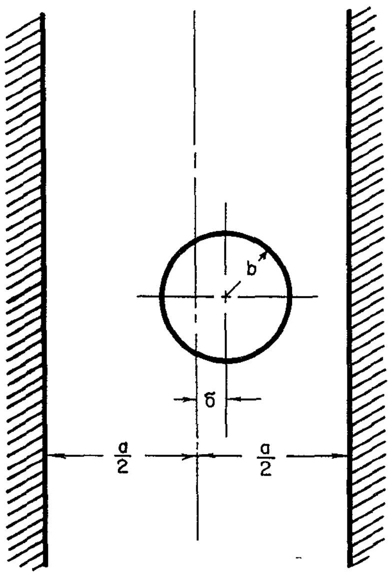

Fig. 2—The cross section of a slab line.

The Variational Expression

The variational expression for the problem is (see Appendix I)

$$\epsilon/C = \frac{\int_0^{2\pi}\int_0^{2\pi} \rho(\phi')G(b, \phi', b, \phi)\rho(\phi)b^2 d\phi' d\phi}{\left[\int_0^{2\pi} \rho(\phi) b\, d\phi\right]^2} \quad (9)$$

which is stationary with respect to small variations in the form of the function $\rho(\phi)$ about its correct form. As outlined in the appendix, to use this method, a trial function containing a number of arbitrary parameters is substituted for $\rho(\phi)$ in (9). The trial function used in this problem is of the form

$$\rho(\phi) = (Q/4\pi b^2)(1 + \alpha_1 \cos\phi + \alpha_2 \cos 2\phi + \cdots \alpha_n \cos n\phi + \cdots + \beta_1 \sin\phi + \beta_2 \sin 2\phi + \cdots \beta_n \sin n\phi). \quad (10)$$

The more terms taken the more accuracy can one expect. Once (10) is substituted in (9) for $\rho(\phi)$ the integrals involved can be evaluated as explicit functions (see Appendix II) of the known parameters a, b, c, and δ (see Fig. 3) and the $2n$ parameters $\alpha_1 \cdots \alpha_n$ and $\beta_1 \cdots \beta_n$. The amplitude factor $Q/4\pi b^2$ cancels out since it appears to the same power both in the numerator and in the denominator of (9).

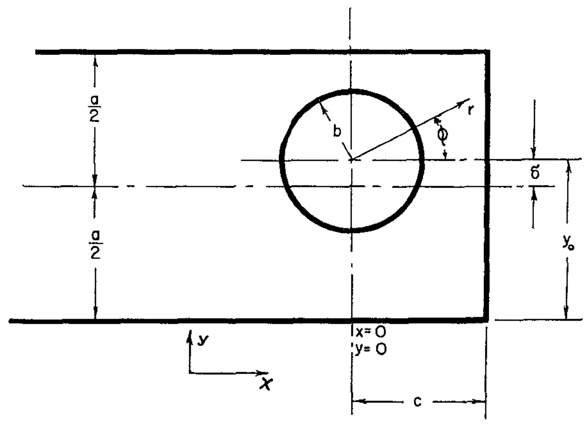

Fig. 3—The coordinates used in describing a trough line.

When the integration has been carried out (9) can be written in the form

$$\frac{\epsilon}{C} = \sum_{j,k=0}^{n} \alpha_j \alpha_k X^{j,k} + 2\sum_{j,k=0}^{n} \alpha_j \beta_k Y^{j,k} + \sum_{j,k=0}^{n} \beta_j \beta_k Z^{j,k} \quad (11)$$

where $\alpha_0 = 1$ and $\beta_0 = 0$ by definition and $X^{j,k}$, $Y^{j,k}$, and $Z^{j,k}$ are the known functions defined in Appendix II and tabulated in Table I which result from the integrations involved in (9) when the trial function of (10) is inserted for $\rho(\phi)$. These turn out to be simple functions of the parameters a, b, c, and δ, and some of the functions used to develop the "second order" approximations are tabulated in Tables I and II.

[4] This is found by assuming $G(x', y', x, y)$ to be expressible as a Fourier series in y and y' with coefficients which are functions of x and x'. The series is substituted in (9) and the resulting series expressed in closed form.

TABLE I

THE FUNCTION

$$X^{j,k} = X^{k,j} = (1/4\pi^2)\int_0^{2\pi}\int_0^{2\pi} \cos(j\phi')\, G(b,\phi',b,\phi)\cos(k\phi)\, d\phi' d\phi$$

j	k	(c/a) infinite	$(c/a)=(3/4)$	$(c/a)=(1/2)$	$(c/a)=(1/4)$
0	1	0.000	$-0.004491(b/a)^2$	$-0.02165(b/a)$	$-0.1086(b/a)$
0	2	$+0.06545(b/a)^2$	$+0.05840(b/a)^2$	$+0.03132(b/a)^2$	$-0.1206(b/a)^2$
1	1	$+0.0397887$ $-0.06545(b/a)^2$	$+0.0397887$ $-0.07250(b/a)^2$	$+0.0397887$ $-0.09958(b/a)^2$	$+0.0397887$ $-0.2515(b/a)^2$
1	2	0.000	$-0.01109(b/a)^3$	$-0.05421(b/a)^3$	$-0.3693(b/a)^3$
2	2	$+0.0198944$ $-0.1130(b/a)^4$	$+0.0198944$ $-0.1305(b/a)^4$	$+0.0198944$ $-0.2010(b/a)^4$	$+0.0198944$ $-1.092(b/a)^4$

TABLE II

SOME TYPICAL FUNCTIONS OF THE TYPE

$$Y^{j,k} = \int_0^{2\pi}\int_0^{2\pi} \cos(j\phi')\, G(b,\phi',b,\phi)\sin(k\phi)\, d\phi' d\phi \quad (c/a) = (1/2)$$

j \ k	0	1	2
0	0.000	$-0.1652(2\delta/a)(b/a)$	$+0.04493(2\delta/a)(b/a)^2$
1	0.000	$+0.04493(2\delta/a)(b/a)^2$	$-0.1836(2\delta/a)(b/a)^3$
2	0.000	$+0.3009(2\delta/a)(b/a)^3$	$+0.05803(2\delta/a)(b/a)^4$

Eq. (11) can be differentiated with respect to the parameters α_j and β_j yielding $2(n+1)$ linear, homogeneous equations of the form

$$0 = \sum_{k=0}^{n} (\alpha_k X^{j,k} + \beta_k Y^{j,k}) \qquad j = 1, 2, \cdots n \tag{12a}$$

$$0 = \sum_{k=0}^{n} (\alpha_k X^{j,k} + \beta_k Z^{j,k}) \qquad j = 1, 2, 3, \cdots n. \tag{12b}$$

When the fact that $\alpha_0 = 1$ and $\beta_0 = 0$ is used these reduce to $2n$ nonhomogeneous equations which can be solved for the α_k's and β_k's by standard methods. Substitution back into (11) then yields an expression for the capacitance C.

PRACTICAL SOLUTIONS

For the results presented here n was taken to be 2 yielding four equations in four unknowns,

$$\alpha_1 X^{1,2} + \alpha_2 X^{1,2} + \beta_1 Y^{1,1} + \beta_2 Y^{1,2} = - X^{1,0} \tag{13a}$$

$$\alpha_1 X^{2,1} + \alpha_2 X^{2,2} + \beta_1 Y^{2,1} + \beta_2 Y^{2,2} = - X^{2,0} \tag{13b}$$

$$\alpha_1 Y^{1,1} + \alpha_2 Y^{2,1} + \beta_1 Z^{1,1} + \beta_2 Z^{2,1} = - Y^{0,1} \tag{13c}$$

$$\alpha_1 Y^{1,2} + \alpha_2 Y^{2,2} + \beta_1 Z^{1,2} + \beta_2 Z^{2,2} = - Y^{0,2}. \tag{13d}$$

It can be shown that all of the functions $Y^{j,k}$ and all of the coefficients β_j are proportional to (δ/a) where δ is the distance by which the cylinder is "off-center" in Fig. 2. This means that, for small values of (δ/a), the third and fourth terms in (13a) and (13b) may be neglected introducing an error in α_1 and α_2 of the order of $(\delta/\alpha)^2$. This approximation becomes exact as δ approaches zero and using it assumes that moving the cylinder "off-center" does not make an appreciable change in the amplitudes of the cosine terms in the series for the charge distribution $\rho(\phi)$.[5] The values of α_1 and α_2 found from (13a) and (13b) can then be substituted into (13c) and (13d) from which β_1 and β_2 can be found using Cramer's rule in terms of a 2×2 determinant. Since (11), which is used to find the capacitance C, is stationary with respect to variations in the parameters α_j and β_j this approximation is valid to within an error of the order of $(\delta/a)^4$.

When the cylinder is exactly centered between the parallel walls of the trough the second order approximation, given by (11) with n taken equal to 2 and using (3) for the relation between the characteristic impedance and the capacitance, is simply

$$Z_0 = {}^{6}Z_{fs}(X^{0,0} + 2\alpha_1 X^{0,1} + 2\alpha_2 X^{0,2} + 2\alpha_1\alpha_2 X^{1,2} + (\alpha_1)^2 X^{1,1} + (\alpha_2)^2 X^{2,2}). \tag{14}$$

where Z_0 is the characteristic impedance of the line in ohms and α_1 and α_2, using the approximations of the preceding paragraph, are given by

$$\alpha_1 = -\frac{\begin{vmatrix} X^{1,0} & X^{1,2} \\ X^{2,0} & X^{2,2} \end{vmatrix}}{\begin{vmatrix} X^{1,1} & X^{1,2} \\ X^{2,1} & X^{2,2} \end{vmatrix}} \tag{15a}$$

$$\alpha_2 = -\frac{\begin{vmatrix} X^{1,1} & X^{1,0} \\ X^{2,1} & X^{2,0} \end{vmatrix}}{\begin{vmatrix} X^{1,1} & X^{1,2} \\ X^{2,1} & X^{2,2} \end{vmatrix}}. \tag{15b}$$

[5] The function $X^{j,k}$, however, changes by order $(\delta/a)^2$ when the cylinder is moved "off-center" and this change must be taken into account because the variational expression is stationary only with respect to variations in the coefficients α_j and β_j and not with respect to changes in the integrals $X^{j,k}$, $Y^{j,k}$, and $Z^{j,k}$.

[6] Z_{fs} is the wave impedance of free space taken in the present work to be 376.735. This is based on Michelson's value (1926) for the velocity of light in free space, 2.99796×10^8 m/s. Other conditions should be adjusted accordingly from the values found from these graphs.

The function $X^{0,0}$ which is the "zero-order" term is just that found by Frankel[2] for a thin wire in a rectangular trough namely

$$X^{0,0} = \frac{1}{2\pi}\log_e\left(\frac{2a}{\pi b}\tanh\frac{\pi c}{a}\right) \tag{16}$$

while the other $X^{j,k}$'s which appear in the formula are given in Table I.

Simplified Formulas

When (15a) and (15b) are substituted into (14) an explicit formula for the characteristic impedance results. Several terms of the initial substitution cancel out and, when the expressions in Table I are used for the $X^{j,k}$'s, (14) becomes

$$Z_0 = Z_{fs}\left[(1/2\pi)\log_e\left(\frac{2a}{\pi b}\tanh\frac{\pi c}{a}\right) + \frac{N_1R + N_2R^2 + N_3R^3}{1 + D_1R + D_2R^2 + D_3R^3}\right] \tag{17a}$$

where $R = (b/a)^2$.

The coefficients N_j and D_j are given in Tables III and IV respectively for different values of the ratio c/a.

TABLE III
THE COEFFICIENTS N_j TO BE USED IN (17) AND (19) FOR THE CALCULATION OF THE SECOND ORDER CORRECTION TERM

(c/a) \ j	1	2	3
1/4	−0.2966	−0.7312	+8.680
1/2	−0.01177	−0.04932	+0.3353
3/4	−0.0005068	−0.1714	+0.3220
infinite	0.000	−0.2153	0.000

TABLE IV
THE COEFFICIENTS D_j TO BE USED IN (17) AND (19) FOR THE CALCULATION OF THE SECOND ORDER CORRECTION TERM

(c/a) \ j	1	2	3
1/4	−6.321	−54.90	+174.8
1/2	−2.503	−10.11	+ 21.58
3/4	−1.822	− 6.559	+ 11.80
infinite	0.000	− 5.682	0.000

For the "slab line" [(c/a) infinite], the numerator and denominator of the second term on the right hand side of (17a) have a common factor and the expression further simplifies to

$$Z_0 = Z_{fs}\left[(1/2\pi)\log_e\left(\frac{2a}{\pi b}\right) - \frac{0.2153R^2}{1 - 5.682R^2}\right] \tag{17b}$$

where, again, $R = (b/a)^2$.

Off-Center Corrections

Perhaps the most interesting result of this investigation is the formula for the change in characteristic impedance which results when the cylinder is moved slightly "off-center" from between the parallel planes.

When δ in Fig. 2 is not equal to zero the coefficients β_1 and β_2 in (13a) to (13d) do not vanish and when the series (11) for ϵ/C is terminated for $j,k=2$, additional terms to those given in (14) result in the expression for ϵ/C all of which, to within $0(\delta/a)^4$ are proportional to $(\delta/a)^2$. In addition to these extra terms all of the $X^{j,k}$'s undergo a change of order $(\delta/a)^2$ also which must be taken into account. This change can be included with the extra terms which result when the cylinder is moved "off-center" allowing the total change in characteristic impedance, which is always less than zero, to be written in the form

$$\Delta Z_0 = -\alpha[(b/a), (c/a)] \times (\delta/a)^2. \tag{18}$$

The parameter α which depends on the ratio b/a and c/a has been calculated for a set of values of these ratios and graphs, sufficiently accurate for purposes of computation, of α vs b/a with c/a as a parameter are given in Fig. 5.

Further Calculations

Tables have been prepared of the functions $X^{j,k}$, $Y^{j,k}$, $Z^{j,k}$ and $\Delta X^{j,k}$ [the change in the value of $X^{j,k}$ when the cylinder is moved off center by an amount δ is $\Delta X^{j,k}\times(\delta/a)^2$] up as far as $j,k=4$ enabling the calculation of a fourth order approximation. All of these functions are simple one or two term polynomials in the ratio (b/a).

A fourth order approximation has been worked out for the "slab line" in the region of fifty ohms. The difference between the second order and fourth order answers was only 0.001 ohms or one part in fifty thousand.

Conclusion

Second Order Formulas

The characteristic impedance of the trough line illustrated in Fig. 1 is given by

$$Z_0 = {}^6Z_{fs}\left[(1/2\pi)\log_e\left(\frac{2a}{\pi b}\tanh\frac{\pi c}{a}\right) + \frac{N_1R + N_2R^2 + N_3R^3}{1 + D_1R + D_2R^2 + D_3R^3}\right] - \alpha[(b/a), (c/a)] \times (\delta/a)^2 \text{ ohms}, \tag{19}$$

where $R = (b/a)^2$ and b is the radius of the cylinder, a is the spacing between the parallel walls of the trough, c is the distance from the center of the cylinder to the bottom of the trough and δ is the distance the center of the cylinder is displaced from the center plane between the parallel plates. The coefficients N_j and D_j are given in Tables III and IV respectively and the coefficient $\alpha(b/a)$, (c/a) is plotted in Fig. 5 as a function of (b/a) with (c/a) as a parameter. The second term on the right hand side of (19) multiplied by $Z_{fs}=376.735$) is plotted in Fig. 4 to a logarithmic scale.

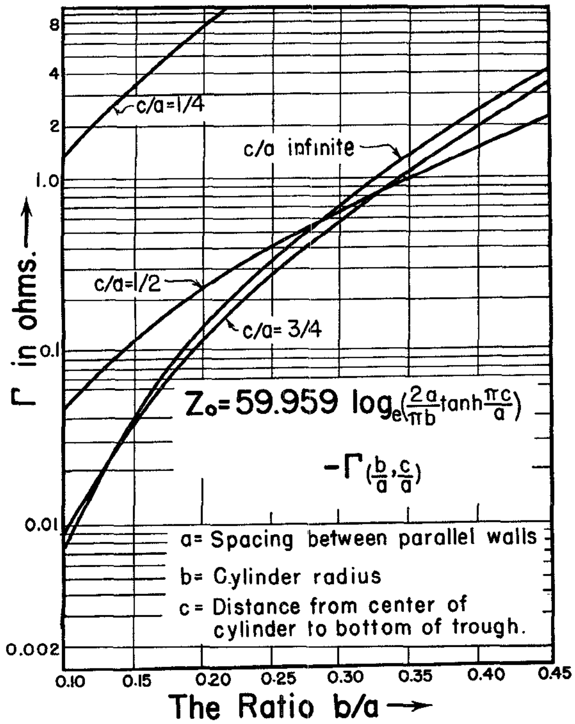

Fig. 4—The second order correction term for the trough line. Value found from this graph are to be subtracted from the zero ordeɐ logarithmic term,

$$59.959 \log_e \left[\left(\frac{2a}{\pi b}\right) \tanh \frac{\pi c}{a}\right].$$

For the "slab line" [$(c/a$ infinite] (19) reduces to

$$Z_0 = {}^6Z_{fs}\left[(1/2\pi)\log_e\left(\frac{2a}{\pi b}\right) - \frac{0.2153R^2}{1+5.682R^2}\right]$$
$$- \alpha(b/a) \times (\delta/a)^2 \text{ ohms} \qquad (20)$$

where $R=(b/a)^2$.

As (b/a) approaches $\frac{1}{2}$ the results break down since the charge distribution $\rho(\phi)$ in the electrostatic model used for calculating ϵ/C approaches the delta function $\delta(\phi \pm (n+\frac{1}{2})\pi)$. The Fourier series for such a distribution has all of its terms of equal magnitude and cutting the series in (10) off at $n=2$ represents a very poor approximation. In practice, however, such large cylinder sizes should be avoided as the capacitance depends very critically on the nature of the surface in the region where the cylinder and the wall almost touch.

Tolerances on the Centering of the Cylinder

If Z_0 is the maximum variation in the characteristic impedance of a trough line which can be tolerated then, from (18), the maximum allowable error in the positioning of the cylinder is given by

$$\delta = a \times \left(\frac{\Delta Z_0}{\alpha}\right)^{1/2} \qquad (21)$$

where α can be found from Fig. 5 as a function of (b/a) and (c/a).

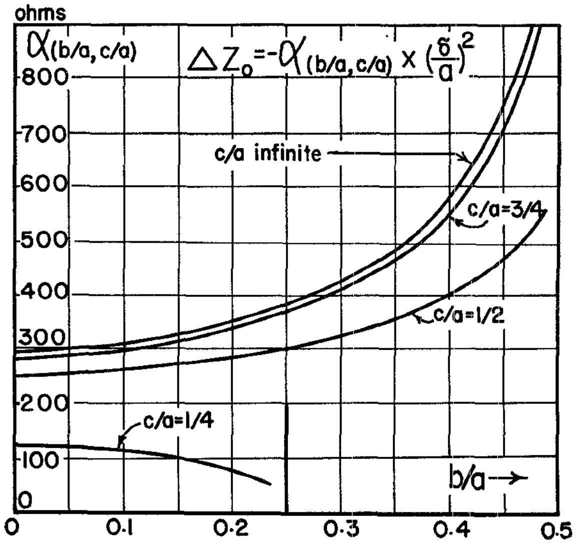

Fig. 5—The "off-center" correction coefficient. The coefficient α found from this graph, when multiplied by $(\delta/a)^2$ where a is the spacing between the parallel planes, gives the correction term to be subtracted from the regular formula for the characteristic impedance whenever the center of the cylinder is off the center line between the parallel planes by an amount δ.

Appendix I

The Variational Expression

The problem under consideration is the solution of an integral equation of the form

$$V_0 = \int_0^{2\pi} \rho(\phi')G(b, \phi', b, \phi)bd\phi'. \qquad (22)$$

Morse and Feshbach[7] treat the general problem of finding a solution to the equation

$$\mathfrak{L}(\psi) = \lambda\mathfrak{M}(\psi) \qquad (23)$$

by a variational method where ψ is the unknown function and $\mathfrak{L}$ and $\mathfrak{M}$ are differential or integral operators. If one sets

$$\psi = \rho(\phi), \qquad (24)$$

$$\mathfrak{L}(\psi) = 1/\epsilon \int_0^{2\pi} \rho(\phi')G(b, \phi', b, \phi)bd\phi' \qquad (25)$$

[7] P. M. Mores and H. Feshbach, "Methods of Theoretical Physics" McGraw-Hill Book Co., Inc., New York, N.Y., 1st ed., vol. 2, p. 1106; 1953.

which is a function of ϕ in the region $0 \leqq \phi \leqq 2\pi$,

$$\mathfrak{M}(\psi) = \int_0^{2\pi} \rho(\phi') b d\phi' \tag{26}$$

and $\lambda = 1/C$ where $C = Q/V_0$, the static capacitance, then (23) becomes

$$1/\epsilon \int_0^{2\pi} \rho(\phi') G(b, \phi', b, \phi) b d\phi' = (1/C) \int_0^{2\pi} \rho(\phi') b d\phi' = V_0 \tag{27}$$

which is the integral equation to be solved. C or $1/C$ is the quantity desired and it is shown by Morse and Feshbach[8] that

$$\delta[\lambda] = \delta\left[\frac{\int \psi \mathfrak{L}(\psi) dV}{\int \psi \mathfrak{M}(\psi) dV}\right] = 0 \tag{28}$$

with respect to arbitrary variations of ψ about its correct value. Interpreting ψ, $\mathfrak{L}$, $\mathfrak{M}$, and λ as in (24)–(26) this yields

$$\delta\left[\frac{1}{C}\right] = \delta\left[\frac{1}{\epsilon} \frac{\int_0^{2\pi} \rho(\phi) \int_0^{2\pi} G(b, \phi, b, \phi') \rho(\phi') b^2 d\phi d\phi'}{\int_0^{2\pi}\int_0^{2\pi} \rho(\phi)\rho(\phi') b^2 d\phi d\phi'}\right] = 0. \tag{29}$$

This means that (9) is stationary with respect to arbitrary variations in the form of $\rho(\phi)$ about its correct form.

To use this method a trial function is put in (9) for $\rho(\phi)$ which contains n arbitrary parameters $\alpha_1 \cdots \alpha_n$. Differentiating with respect to these parameters and equating the results to zero leads to n homogeneous, linear equations in the n unknown α's.[9]

Appendix II

Evaluating the Integrals Involved in the Variational Method

When $\rho(\phi)$ in (9) is replaced by the series of (10) and use is made of the orthogonality properties of the trigonometric functions to give $4\pi^2 b^2$ for the denominator of the right hand side of (9), ϵ/C can be expressed by the series in (11) where integrals of the following type are encountered

[8] *Ibid.*, p. 1109 (9.4.8).
[9] *Ibid.*, p. 1107.

$$X^{j,k} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos(j\phi) G(b, \phi', b, \phi) \cos(k\phi') d\phi d\phi' \tag{30}$$

$$Y^{j,k} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos(j\phi) G(b, \phi', b, \phi) \sin(k\phi') d\phi d\phi' \tag{31}$$

$$Z^{j,k} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \sin(j\phi) G(b, \phi', b, \phi) \sin(k\phi') d\phi d\phi'. \tag{32}$$

These integrals depend on the parameters j and k and, through the Green's function $G(b, \phi', b, \phi)$, on the dimensions a, b, and c of the transmission line.

These integrals can be evaluated by removing the singularity from $G(r', \phi', r, \phi)$ at $r = r'$, $\phi = \phi'$ and using the following lemma. *If* $\nabla^2\psi(r, \varphi) = 0$

$$\int_0^{2\pi} \psi(r, \phi) e^{in\phi} d\phi = (\pi/n!) r^{|n|} \underset{r\to 0, \vartheta \to 0}{\text{Limit}} \frac{\partial^n}{\partial r^n}\left[1 + (i/n)\frac{\partial}{\partial \phi}\right]\psi(r, \phi) \quad \text{for } n \neq 0,$$
$$= 2\pi \underset{r\to 0, \vartheta\to 0}{\text{Limit}} \psi(r, \phi) \qquad n = 0. \tag{33}$$

The differentiations involved, if carried out directly, are rather messy. However it has been found possible to express the results of this lemma in terms of a set of functions, represented by a finite power series, the coefficients of which obey a simple repetition formula.

Putting

$$\xi_0^n(z) = \frac{e^{inz}}{1 - e^{inz}} \tag{34}$$

and

$$(in)^k \xi_k^n(z) = \frac{d^k}{d_z^k} \xi_0^n(z) \tag{35}$$

it can be shown that

$$\xi_k^n(z) = \sum_{m=1}^{k} c_m^k [\xi_0^n(z)]^m (1 + \xi_0^n(z)), \tag{36}$$

where the coefficients c_m^k obey the repetition formula

$$c_1^k = 1; \quad c_k^k = k!; \quad \text{and} \quad c_m^k = m(c_{m-1}^{k-1} + c_m^{k-1}).$$

For special values of the argument z, (36) is of a particularly simple form. For example, one of the functions which occurs in applying this lemma is

$$\xi_{j+k-1}^{(\pi/a)}(a) = \sum_{m=1}^{j+k-1} (-1)^m \frac{c_m^{j+k+1}}{2^{m+1}}. \tag{37}$$

It is beyond the scope of this paper to go into the details of these functions beyond pointing out, as seen from Table I, that integrals of the types under consideration, when evaluated in this way, come out as simple polynomials in the ratio (b/a).